arXiv:gr-qc/0308006v1 1 Aug 2003

Singular Hypersurfaces in Einstein-Gauss-Bonnet Theory of Gravitation

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Abstract

We present a general formalism for describing singular hypersurfaces in the Einstein theory of gravitation with a Gauss-Bonnet term. The junction conditions are given in a form which is valid for the most general embedding and matter content and for coordinates chosen independently on each side of the hypersurface. The theory is applied to both a time-like and a light-like hypersurface in brane-cosmology.

PACS number(s): 04.20.Cv

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Space–times with noncompact extra dimensions are presently extensively studied, in particular in the context of brane-cosmology. These models, which are inspired from high-energy physics, string theory and M-theory, lead to a brane-world picture. In the most popular model [1]-[2] our ordinary four-dimensional space–time is the history of a three-dimensional brane in a five-dimensional space–time (the bulk). All matter and gauge fields, except gravity, are confined to a 3-brane whose history is a time–like shell embedded in a 5-dimensional anti-de Sitter space-time. An additional feature of string theory is that the low-energy effective action contains terms which are quadratic in the curvature. The Gauss-Bonnet term, $L_{GB} = R_{\alpha\beta\rho\sigma}\, R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta}\, R^{\alpha\beta} + R^2$, has then been included in the Einstein-Hilbert action.

When the Gauss-Bonnet term is introduced into the brane-world models the field equations of the 5-D space-time are modified and one also expects the field equations on the brane to be modified. Several authors have studied this problem [3] and have for the most part only considered the particular situation corresponding to brane-cosmology (Z_2-symmetry for the embedding, anti-de Sitter geometry for the bulk and perfect fluid on the brane). Their results concerning the modifications of the Friedmann equation on the brane are still controversial. Consequently it is important to have a systematic treatment of the junction conditions involved. The approach taken by some of the authors in [3] uses a Lagrangian with appropriate boundary terms (the Lagrangian method for shells in General Relativity is not free of ambiguities– see for example [4]) whereas our approach starts with the field equations and makes use of the Gauss–Codazzi equations. *It is the purpose of this paper to present a general formalism for the description of the junction-conditions on a singular hypersurface when the Gauss-Bonnet term is present.* Our formalism places no restrictions on the matter content or the geometry of the outer space or on the matter content of the shell. The only restriction is the basic embedding condition that the induced metrics on the hypersurface coincide. This is achieved by extending to the Einstein-Gauss-Bonnet theory a previous work [5] where the junction-conditions in the Einstein theory were given for an arbitrary hypersurface (time–like, space–like or light–like). Our general formalism shows that, in spite of the presence of quadratic terms in the curvature, no regularisation of the Dirac δ-function is required and a well–defined set of junction conditions is obtained. We then apply our results to brane-world cosmology and consider both a time–like and a null hypersurface.

Our results, summarized in (1.13)–(1.17) below, are completely general and do not assume any symmetries (of the space–time or the embedding) in contradistinction to the works cited in [3] where, in particular Z_2-symmetry

is introduced at the outset. This is a source of confusion with regard to the distinction between *the average* and *the jump* of a quantity which is discontinuous across the hypersurface (equations such as (1.10) and (1.15) below contain both averages and jumps of quantities). This appears to be the fundamental origin of the discrepancy between our results and those cited in [3].

We consider a 5-dimensional space–time $\mathcal{M}$ with a system of local coordinates $\{x^\mu\}$, $\mu = 0, 1, 2, 3, 4$. The components of tensors on $\mathcal{M}$ in this coordinate system will be identified by an index 5. For example the metric tensor components will be denoted ${}^5g_{\mu\nu}$. The field equations are

$$ {}^5G_{\mu\nu} + \Lambda_5\ {}^5g_{\mu\nu} + 2\alpha H_{\mu\nu} = \kappa_5\ {}^5T_{\mu\nu}\ , \tag{0.1} $$

where ${}^5G_{\mu\nu}$ is the Einstein tensor calculated with the metric tensor ${}^5g_{\mu\nu}$, Λ_5 is the cosmological constant, α is a coupling constant and $H_{\mu\nu}$ is the Lovelock tensor which is given by

$$ \begin{aligned} H_{\mu\nu} &= {}^5R\ {}^5R_{\mu\nu} - 2\ {}^5R_\mu{}^\lambda\ {}^5R_{\lambda\nu} - 2\ {}^5R^{\alpha\beta}\ {}^5R_{\alpha\mu\beta\nu} + {}^5R_{\mu\rho\kappa\lambda}\ {}^5R_\nu{}^{\rho\kappa\lambda} \\ &- \frac{g_{\mu\nu}}{4}({}^5R_{\alpha\beta\rho\sigma}\ {}^5R^{\alpha\beta\rho\sigma} - 4\ {}^5R_{\alpha\beta}\ {}^5R^{\alpha\beta} + {}^5R^2)\,. \end{aligned} \tag{0.2} $$

Here ${}^5R_{\mu\rho\kappa\lambda}$, ${}^5R_{\mu\nu}$, ${}^5R$ are the components of the Riemann tensor, Ricci tensor and Ricci scalar respectively calculated with the metric tensor ${}^5g_{\mu\nu}$. In the right hand side of the field equations the coefficient κ_5 is the 5-D gravitational constant and ${}^5T_{\mu\nu}$ is the stress-energy tensor describing the matter content (of the bulk and the brane in the brane-world language).

The space–time manifold $\mathcal{M}$ is divided into two domains $\mathcal{M}^\pm$ by a singular hypersurface $\mathcal{N}$ on which the metric tensor is only C^0. As a consequence of this the Riemann curvature tensor contains a Dirac δ-term with support on $\mathcal{N}$. Each domain $\mathcal{M}^\pm$ admits a metric tensor ${}^5g^\pm$ and all quantities referring to $\mathcal{M}^\pm$ will be denoted by an index $\pm$. We denote the jumps across $\mathcal{N}$ of a quantity $F^\pm$ defined on $\mathcal{M}^\pm$ by $[F] = F^+|_\mathcal{N} - F^-|_\mathcal{N}$, where $|_\mathcal{N}$ indicates that $F^\pm$ is to be evaluated on the $\pm$ sides of $\mathcal{N}$ respectively.

1 Junction conditions on a timelike or spacelike hypersurface

Let $\{x^\mu\}$ be a local coordinate system covering both sides of the hypersuface $\mathcal{N}$ in terms of which the components of the metric tensor are continuous across $\mathcal{N}$. Let $\Phi(x) = 0$ be the equation of $\mathcal{N}$ in these coordinates, with $\Phi > 0 (< 0)$ in $\mathcal{M}^+(\mathcal{M}^-)$. Greek indices take values $0, 1, 2, 3, 4$ and the

components of the metric tensor are $g^{\pm}_{\mu\nu}$ in $\mathcal{M}^{\pm}$ respectively. If $F^{\pm}(x)$ are two quantities (the components of a tensor for example) defined on $\mathcal{M}^{\pm}$ respectively, we define the hybrid quantity $\tilde{F}$ by

$$\tilde{F}(x) = F^{+}\,\Theta(\Phi) + F^{-}\,(1 - \Theta(\Phi))\ , \tag{1.1}$$

where $\Theta(\Phi)$ is the Heaviside step function which is equal to unity (zero) when $\Phi > 0 (< 0)$. Thus in particular for the metric tensor we have ${}^5\tilde{g}_{\mu\nu}$ defined and since the metric is continuous across $\mathcal{N}$ we can write $[{}^5g_{\mu\nu}] = 0$. As a result of the definition (1.1) we have for two quantities $F^{\pm}$ and $G^{\pm}$ the product rule, $\tilde{F}\tilde{G} = \widetilde{FG}$, because $\Theta(\Phi)(1-\Theta(\Phi))$ vanishes distributionally. The normal to the hypersurface has components

$$n_\mu = \chi^{-1}(x)\,\partial_\mu\Phi(x)\,, \tag{1.2}$$

where $\chi(x)$ is a normalizing factor such that

$$n\cdot n = {}^5g^{\mu\nu}n_\mu n_\nu\,|_{\pm} = \epsilon\,, \tag{1.3}$$

with $\epsilon = +1\,(-1)$ if the hypersurface is time–like (space–like). With these definitions the partial derivative of $\tilde{F}$ takes the form

$$\partial_\mu\tilde{F} = \widetilde{\partial_\mu F}\; + \;[F]\,\chi n_\mu\,\delta(\Phi)\,. \tag{1.4}$$

A singular term proportional to the Dirac δ-funtion appears whenever F is discontinuous across the hypersurface $\mathcal{N}$. The metric and its tangential derivatives are continuous across $\mathcal{N}$ but its transverse derivatives are not. To characterize the discontinuities in the transverse derivatives of the metric tensor we define the symmetric tensor $\gamma_{\mu\nu}$ by

$$[\partial_\alpha\,{}^5g_{\mu\nu}] = \epsilon\,n_\alpha\,\gamma_{\mu\nu}\,. \tag{1.5}$$

The tensor $\gamma_{\mu\nu}$ is only defined *on* $\mathcal{N}$ and its projection onto $\mathcal{N}$ is unique. Thus $\gamma_{\mu\nu}$ is free up to the gauge transformation, $\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + v_\mu n_\nu + n_\mu v_\nu$, where v is an arbitrary vector field on $\mathcal{N}$. This gauge freedom can always be used to have $\gamma_{\mu\nu}n^\nu = 0$ whenever $\mathcal{N}$ is not lightlike. Using (1.5) we find that the Christoffel symbols satisfy ${}^5\Gamma^\lambda_{\mu\nu} = {}^5\tilde{\Gamma}^\lambda_{\mu\nu}$ and $[{}^5\Gamma^\lambda_{\mu\nu}] = \epsilon\,\gamma^\lambda_{(\mu}n_{\nu)} - \frac{\epsilon}{2}\gamma_{\mu\nu}n^\lambda$, with round brackets around indices denoting symmetrisation. Then using (1.4) the Riemann curvature tensor ${}^5R_{\kappa\lambda\mu\nu}$ can be decomposed into the sum of a tilde-term defined as in (1.1) and a term containing a Dirac δ-function:

$${}^5R_{\kappa\lambda\mu\nu} = {}^5\tilde{R}_{\kappa\lambda\mu\nu} + \hat{R}_{\kappa\lambda\mu\nu}\,\epsilon\chi\delta(\Phi)\ , \tag{1.6}$$

where

$$\hat{R}_{\kappa\lambda\mu\nu} = 2\,n_{[\kappa}\gamma_{\lambda][\mu}n_{\nu]}\ . \tag{1.7}$$

The square brackets here around indices denote skew–symmetrisation. A similar decomposition therefore exists for the Ricci tensor ${}^5R_{\mu\nu}$, the Ricci scalar ${}^5R$ and the Einstein tensor ${}^5G_{\mu\nu} = {}^5R_{\mu\nu} - \frac{1}{2}\,{}^5R\,{}^5g_{\mu\nu}$. The singular part of the Einstein tensor is given by

$$\hat{G}_{\mu\nu} = \gamma_{(\mu}n_{\nu)} - \frac{\gamma}{2}\,n_\mu\,n_\nu - \frac{\gamma^\dagger}{2}\;{}^5g_{\mu\nu} - \frac{\epsilon}{2}\,(\gamma_{\mu\nu} - \gamma\;{}^5g_{\mu\nu})\ , \tag{1.8}$$

where we have introduced $\gamma \equiv \gamma^\mu_\mu$, $\gamma_\mu \equiv \gamma_{\mu\nu}\,n^\nu$ and $\gamma^\dagger \equiv \gamma_\mu\,n^\mu$ (recall that one can always choose the gauge such that $\gamma_\mu = \gamma^\dagger = 0$).

If we now use (1.6) to calculate $H_{\mu\nu}$ in (0.2) undefined δ^2-terms will appear since $H_{\mu\nu}$ is quadratic in the Riemann tensor. However one can show that these terms simply disappear. To see this we first note that for any tensor A having the form $A = \tilde{A} + \hat{A}\,\chi\,\epsilon\,\delta(\Phi)$ and any other tensor B having the same form their product AB contains a δ^2-term with coefficient $\hat{A}\hat{B}\chi^2\epsilon^2$. If this is applied to the calculation of $H_{\mu\nu}$ and use is made of the expressions for $\hat{R}$, $\hat{R}_{\mu\nu}$ and $\hat{R}_{\kappa\lambda\mu\nu}$ given or derived from (1.7) we find that the total contribution of all products of the type $\hat{A}\hat{B}$ is zero. Therefore the coefficient of the δ^2 term in $H_{\mu\nu}$ vanishes and we can write

$$H_{\mu\nu} = \tilde{H}_{\mu\nu} + \hat{H}_{\mu\nu}\,\epsilon\chi\,\delta(\Phi)\ , \tag{1.9}$$

where $\hat{H}_{\mu\nu}$ is given by

$$\begin{aligned}\hat{H}_{\mu\nu} &= \hat{R}_{\mu\nu}\;{}^5\bar{R} + \hat{R}\;{}^5\bar{R}_{\mu\nu} - 2\,(\hat{R}^{\alpha\beta}\;{}^5\bar{R}_{\alpha\mu\beta\nu} + \hat{R}_{\alpha\mu\beta\nu}\;{}^5\bar{R}^{\alpha\beta})\\ &+\hat{R}_{\mu\kappa\rho\lambda}\;{}^5\bar{R}_\nu{}^{\kappa\rho\lambda} + \hat{R}_\nu{}^{\kappa\rho\lambda}\;{}^5\bar{R}_{\mu\kappa\rho\lambda} - 2\,(\hat{R}_\mu{}^\lambda\;{}^5\bar{R}_{\nu\lambda} + \hat{R}_\nu{}^\lambda\;{}^5\bar{R}_{\mu\lambda})\\ &-\frac{{}^5g_{\mu\nu}}{2}(\hat{R}_{\alpha\beta\rho\sigma}\;{}^5\bar{R}^{\alpha\beta\rho\sigma} - 4\,\hat{R}_{\alpha\beta}\;{}^5\bar{R}^{\alpha\beta} + \hat{R}\;{}^5\bar{R})\ .\end{aligned} \tag{1.10}$$

In those expressions $\tilde{H}_{\mu\nu}$ has the general form (1.1) and the bar denotes the 'average' of a quantity which is discontinuous across $\mathcal{N}$ (thus $\bar{A} = (A^+|_\mathcal{N} + A^-|_\mathcal{N})/2$ for any A for which $[A] \neq 0$). Also the property $\Theta(x)\,\delta(x) = \frac{1}{2}\,\delta(x)$, which is distributionally valid, has been used. When these results are substituted into the field equations (0.1) we find that the stress-energy tensor on the right hand side of the equations is decomposed into a tilde term and a singular term with the latter indicating the presence, in general, of a thin shell with history $\mathcal{N}$. Thus

$${}^5T_{\mu\nu} = {}^5\tilde{T}_{\mu\nu} + S_{\mu\nu}\chi\delta(\Phi)\,, \tag{1.11}$$

where the tensor $S_{\mu\nu}$ is the surface stress-energy tensor of the shell. Identifying the singular terms on each side of the field equations we have

$$\kappa_5\, S_{\mu\nu} = \epsilon\, \hat{G}_{\mu\nu} + 2\, \alpha\, \epsilon\, \hat{H}_{\mu\nu}\ , \tag{1.12}$$

where $\hat{G}_{\mu\nu}$ is given by (1.8) and $\hat{H}_{\mu\nu}$ by (1.9). Since (1.8) and (1.10) lead to $\hat{G}_{\mu\nu}\, n^\nu = 0$ and $\hat{H}_{\mu\nu}\, n^\nu = 0$ we also have $S_{\mu\nu}\, n^\nu = 0$. This tangential property for $S_{\mu\nu}$, and the vanishing of the δ^2-terms in $H_{\mu\nu}$ are both a direct consequence of the particular form of the Gauss-Bonnet term.

The Eq.(1.12) represents the junction conditions on $\mathcal{N}$ in a system of coordinates covering both sides of the hypersurface. This presentation is equivalent to the usual Israel junction conditions formalism [6] as there is a direct relation between the tensor $\gamma_{\mu\nu}$ and the jump in the extrinsic curvature of $\mathcal{N}$. The formalism based on the extrinsic curvature has the advantage of allowing the 5-D space–time coordinates to be chosen freely and independently on each side of the hypersurface. Let $\{x^\mu_\pm\}$ be local chartss for the domains $\mathcal{M}^\pm$ and introduce on the hypersurface $\mathcal{N}$ four intrinsic coordinates $\{\xi^a\}$ with $a = 0, 1, 2, 3$. The four holonomic tangent basis vectors $e_{(a)} = \partial/\xi^a$ have components $e^\mu_{(a)}|_\pm = \partial x^\mu_\pm/\partial\xi^a$, and the induced metrics $g_{ab} = {}^5g_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)}|_\pm$ match on $\mathcal{N}$. The extrinsic curvature is defined by $K_{ab} = -n_\mu\, e^\mu_{(a)|\lambda}\, e^\lambda_{(b)}$ and takes different values $K^\pm_{ab}$ on each side of $\mathcal{N}$. Here the stroke denotes covariant differentiation associated with the five dimensional metric tensor on either side of $\mathcal{N}$. It can be shown that $\gamma_{ab} \equiv 2[K_{ab}]$ is the projection on $\mathcal{N}$ of the tensor $\gamma_{\mu\nu}$ introduced in (1.5), thus $\gamma_{ab} = \gamma_{\mu\nu}\, e^\mu_{(a)}\, e^\nu_{(b)}$. The projection of the surface stress–energy tensor onto $\mathcal{N}$ is given by $S_{ab} = S_{\mu\nu}\, e^\mu_{(a)}\, e^\nu_{(b)}$. In similar fashion the projections $\hat{G}_{ab}$ and $\hat{H}_{ab}$ onto $\mathcal{N}$ of the singular parts of $G_{\mu\nu}$ and $H_{\mu\nu}$ are defined. The junction conditions (1.12) now read

$$\kappa_5\, S_{ab} = \epsilon \hat{G}_{ab} + 2\, \alpha\, \epsilon \hat{H}_{ab}\ . \tag{1.13}$$

Using the Gauss-Codazzi equations and their contracted forms in (1.8) and (1.10) we find that

$$\epsilon\, \hat{G}_{ab} = -[K_{ab}] + [K]\, g_{ab}\ , \tag{1.14}$$

$$\epsilon\, \hat{H}_{ab} = -2\, [K^{cd}]\, ({}^*R^*_{acbd} - {}^*\bar{r}^*_{acbd})\ . \tag{1.15}$$

In the last relation ${}^*R^*_{acbd}$ is the left and right dual of the intrinsic Riemann tensor of the hypersurface $\mathcal{N}$ and is given by

$${}^*R^*_{acbd} = -R_{acbd} + 2\, g_{a[b}R_{d]c} - 2\, g_{c[b}R_{d]a} - R\, g_{a[b}g_{d]c}\ . \tag{1.16}$$

The final term in (1.15) is the left and right dual of the average on $\mathcal{N}$ of

$$r_{acbd} = K_{ab}K_{cd} - K_{ad}K_{bc}\ , \tag{1.17}$$

which has the same algebraic symmetries as R_{acbd}. When the Gauss-Bonnet term is absent ($\alpha = 0$) only $\hat{G}_{ab}$, which simply depends on $[K_{ab}]$, appears in the junction relations and we recover the usual Israel conditions [6]. This limit $\alpha = 0$ is not in, for example, the cited work of Davis [3], if this work is interpreted using the standard convention for the normal to be found in [5] or [6]. The Gauss–Bonnet contribution to the junction relations is contained in $\hat{H}_{ab}$ which depends not only on $[K_{ab}]$, but also on the intrinsic curvature tensor R_{acbd} of $\mathcal{N}$ and on the average of products of the extrinsic curvature. The equation (1.13) describes the evolution of the shell once a choice is made of the surface stress-energy tensor S_{ab}. In brane-cosmology it provides the new field equations on the brane. Two other relations can be derived by considering the jump of the field equations (0.1)-(0.2) across $\mathcal{N}$ contracted with $n^\mu\, e^\nu_{(a)}$ or $n^\mu\, n^\nu$, and making use of the Gauss-Codazzi equations:

$$S^b{}_{a;b} = -[{}^5T_{\mu\nu}\, n^\mu\, e^\nu_{(a)}] + \frac{2\,\alpha}{\kappa_5}\left(\epsilon \hat{H}^b{}_{a;b} + [H_{\mu\nu}\, n^\mu\, e^\nu_{(a)}]\right) \;, \tag{1.18}$$

$$S_{ab}\, \bar{K}^{ab} = [{}^5T_{\mu\nu}\, n^\mu\, n^\nu] - \frac{\epsilon}{\kappa_5}[\Lambda_5] + \frac{2\,\alpha}{\kappa_5}\left(\epsilon \hat{H}_{ab}\, \bar{K}^{ab} - [H_{\mu\nu}\, n^\mu\, n^\nu]\right) \;. \tag{1.19}$$

where the semicolon here denotes the covariant derivative associated with g_{ab}. For a time–like shell the $a = 0$ component of (1.18) gives the energy conservation equation.

As an illustration of our general formalism we consider brane-cosmology and the most popular case where the 5-D space–times have anti-de Sitter geometry, the embedding is Z_2 symmetric and the matter on the brane is a perfect fluid with proper–density ρ and pressure p. In this case $\epsilon = 1$ as $\mathcal{N}$ is taken to be time–like. The assumption of Z_2 symmetry simplifies the expression for the average of the tensor r_{acbd} defined above to read

$$\bar{r}_{acbd} = \frac{1}{4}\,\left([K_{ab}]\,[K_{cd}] - [K_{ad}]\,[K_{bc}]\right) \;. \tag{1.20}$$

The 5-dimensional metric is given via the line–element

$$ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\chi^2 + r^2 f_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \;. \tag{1.21}$$

Here $f_k(\chi) = 1,\ \sin\chi,\ \sinh\chi$ for $k = 0,\ 1,\ -1$, and $h(r)$ is given by [7]

$$h(r) = k + \frac{r^2}{4\alpha}\,(1 - \sqrt{A(r)}) \;, \tag{1.22}$$

where $A(r) = 1 + \frac{4}{3}\,\alpha\,\Lambda_5 + 8\,\alpha\,\frac{m}{r^4}$, with m being a mass parameter. The shell is radially moving with the law of motion $r = a(\tau),\ t = t(\tau)$, and τ given by

$dt/d\tau = (h + \dot{a}^2)^{1/2}/h$. The two functions $f_k(\chi)$ and $h(r)$ are the same on both sides of $\mathcal{N}$ and the metric on $\mathcal{N}$ has the Robertson–Walker form

$$ds^2|_{\mathcal{N}} = -d\tau^2 + a^2(\tau)\, d\chi^2 + a^2(\tau)\, f_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \qquad (1.23)$$

Now S^{ab} has the perfect fluid form $S^{ab} = (\rho + P)\, u^a\, u^b + P\, g^{ab}$ with $u^a = (1,0,0,0)$ the four-velocity of the fluid. The assumption of Z_2 symmetry implies for the extrinsic curvature $K^+_{ab} = -K^-_{ab} \equiv K_{ab}$, and because the metric given by (1.21) is spherically symmetric we have $K^\chi_\chi = K^\theta_\theta = K^\phi_\phi = \zeta\sqrt{\dot{a}^2 + h(a)}/a$, where $\dot{} = d/d\tau$ and $\zeta = \text{sign}\,(n^\mu\, \partial_\mu r)$. Introducing these properties into the equation (1.13) contracted twice with u^a we arrive at

$$\kappa_5\, \rho = -\frac{6\zeta}{a}\sqrt{\{\dot{a}^2 + h(a)\}\, A(a)}\ . \qquad (1.24)$$

Squaring this we obtain the Friedmann equation on the brane when a Gauss-Bonnet term is present

$$\mathcal{H}^2 = -\frac{k}{a^2} + \frac{\kappa_5^2\, \rho^2}{36A(a)} - \frac{1}{4\alpha}\{1 - \sqrt{A(a)}\}\ . \qquad (1.25)$$

Here $\mathcal{H} = \dot{a}/a$ is the Hubble parameter. It is easy to check that in the limit $\alpha \to 0$ this equation reduces to the usual Friedmann equation on the brane without the Gauss-Bonnet term (see for example [2]).

2 Junction conditions for a light–like hypersurface

The general formalism that we have developed for a time–like or a space–like shell can be adapted to the case of a null shell. We briefly present the main results here and refer the reader to [5] for more details concerning the properties of a light–like shell. As an illustration we describe a null brane propagating in the brane-cosmological model already considered in the previous section.

The hypersurface $\mathcal{N}$ is now light–like and therefore its normal n satisfies (1.3), with $\epsilon = 0$, and is tangent to $\mathcal{N}$. The definition (1.2) for n still applies but the function $\chi(x)$ is now arbitrary. We introduce a transversal vector N on $\mathcal{N}$ such that $N \cdot n = \eta^{-1} \neq 0$, and describe the discontinuity across $\mathcal{N}$ in the transverse first derivative of the metric by the tensor $\gamma_{\mu\nu}$ which is defined by the same equation (1.5) with ϵ replaced by η. The decomposition (1.6) of the Riemann tensor still applies with again ϵ replaced by η and its

singular part has the same expression (1.7). However the singular part of the Einstein tensor (1.8) now specializes to only the first three terms in (1.8). Note that here the gauge freedom for $\gamma_{\mu\nu}$ cannot be used to make $\gamma_\mu = 0$. It can be shown that δ^2-terms are still absent in $H_{\mu\nu}$ and that the equations (1.9)-(1.12) with ϵ replaced by η are still valid. It can also be checked that the tangential property $S_{\mu\nu}\, n^\nu = \hat{G}_{\mu\nu}\, n^\nu = \hat{H}_{\mu\nu}\, n^\nu = 0$ still applies.

As in the previous section we introduce on $\mathcal{N}$ four intrinsic coordinates $\{\xi^a\}$ with $a = 1, 2, 3, 4$ and the corresponding holonomic tangent vectors $e_{(a)} = \partial/\partial\xi^a$. We choose here $e_{(1)} = n$ future-directed and therefore the other vectors $e_{(A)}$ with $A = 2, 3, 4$ are space–like. We then define on $\mathcal{N}$ the basis $\{N, n, e_{(A)}\}$ where the transversal is chosen light–like, perpendicular to the $e_{(A)}$'s and oriented toward the future of $\mathcal{N}$. Thus N satisfies $N \cdot N = N \cdot e_{(A)} = 0$ and $N \cdot n = \eta^{-1} = -1$. The induced metric $g_{ab} = e_{(a)} \cdot e_{(b)}|_\pm$ on $\mathcal{N}$ is degenerate and reduces to $g_{AB} = e_{(A)} \cdot e_{(B)}|_\pm$.

Of the following two tensors, $K_{ab} = -n_\mu\, e^\mu_{(a)|\lambda}\, e^\lambda_{(b)}$, $\mathcal{K}_{ab} = -N_\mu\, e^\mu_{(a)|\lambda}\, e^\lambda_{(b)}$, where the stroke has the same meaning as in section 1, only the second tensor describes transverse properties and represents an extrinsic curvature. While K_{ab} is purely intrinsic and therefore continuous across $\mathcal{N}$, $\mathcal{K}_{ab}$ is discontinuous across $\mathcal{N}$ with a jump described by $\gamma_{ab} \equiv \gamma_{\mu\nu}\, e^\mu_{(a)}\, e^\nu_{(b)} = 2[\mathcal{K}_{ab}]$. If we express the tensors $S^{\mu\nu}$, $\hat{G}^{\mu\nu}$ and $\hat{H}^{\mu\nu}$ in terms of the tangent basis $\{e_{(a)}\}$, and thus write $S^{\mu\nu} = S^{ab} e^\mu_{(a)} e^\nu_{(b)}$, with similar expressions holding for $\hat{G}^{\mu\nu}$ and $\hat{H}^{\mu\nu}$, then the equation for $\hat{G}^{ab}$ is (see the eq.(31) of [5])

$$2\,\hat{G}^{ab} = -(\gamma_{cd}\, g^{cd}_*)\, n^a n^b \,-\, \gamma^\dagger\, g^{ab}_* \,+\, \{g^{ac}_*\, n^b n^d \,+\, g^{bc}_*\, n^a n^d\}\gamma_{cd}\,, \tag{2.1}$$

where one takes for g^{ab}_* the matrix g^{AB}, inverse of g_{AB}, bordered by zeros. The equation for $\hat{H}^{ab}$ is derived from the Gauss-Codazzi equations written on the basis $\{N, n, e_{(A)}\}$. It leads to a complicated expression which contain the intrinsic curvature tensor R_{abcd}, the intrinsic three-dimensional curvature K_{AB}, and the extrinsic curvature $\mathcal{K}_{ab}$. It can be shown [5],[8] that a null shell and an impulsive gravitational generally coexist with the hypersurface $\mathcal{N}$ representing their space–time history.

As an illustration of this theory we consider a spherical null shell propagating radially in the brane-cosmological model of section 2. It is convenient to rewrite the line–element (1.21) in terms of the Eddington retarded or advanced time coordinate u. We introduce the constant sign factor ζ which is $+1(-1)$ if the light cone $\mathcal{N}$ with equation $u = \text{const.}$ is expanding (contracting) towards the future. The embedding requires that the function $f_k(\chi)$ be the same on both sides of $\mathcal{N}$, but the cosmological constant $\Lambda_5^\pm$ and the mass parameter $m_\pm$ can differ. The components of the normal n and the transversal N are $n^\mu = \zeta\, \delta^\mu_r$ and $N^\mu = (1, -\zeta\, h(r)/2, 0, 0, 0)$, and the only

non–vanishing components of $\mathcal{K}_{ab}$ and K_{ab} are $\mathcal{K}^{\chi}_{\chi} = \mathcal{K}^{\theta}_{\theta} = \mathcal{K}^{\phi}_{\phi} = -\zeta\, h/2\, r$ and $K^{\chi}_{\chi} = K^{\theta}_{\theta} = K^{\phi}_{\phi} = \zeta/r$. Because $\mathcal{N}$ is a null cone, and therefore the light–like signal is spherical–fronted, $\mathcal{N}$ cannot be the history of an impulsive gravitational wave. In fact $\mathcal{N}$ is the history of a spherical null shell which is simply characterized by its surface energy density ρ and surface pressure P. In other words the surface stress-energy tensor S^{ab} has components $S^{11} = \rho$, $S^{AB} = P\, g^{AB}$ and $S^{A1} = 0$ with

$$\kappa_5\, \rho = \frac{3\,\zeta}{2\,r}\,[h(r)] - \frac{2\,\alpha\,\zeta}{r^3}\,[h(r)]\left\{\frac{15}{2}\,\bar{h}(r) - \frac{(1-3k\, f_k(\chi)^2)}{f_k(\chi)^2}\right\}\ , \tag{2.2}$$

$$\kappa_5\, P = -\frac{9\,\alpha\,\zeta}{r^3}\,[h(r)]\left(1+\frac{\bar{h}(r)}{2}\right)\ . \tag{2.3}$$

When the Gauss-Bonnet term is absent ($\alpha = 0$) the surface pressure vanishes ($P = 0$) and the surface energy density of the null shell is given simply by

$$\kappa_5\rho = -\frac{3\,\zeta}{2\,r^3}\,[m]\ -\frac{\zeta\, r}{4}\,[\Lambda_5]\ . \tag{2.4}$$

This is the analogue in 5-D of the known expression for a similar null shell propagating in the four-dimensional Schwarzshild-(anti)de Sitter spacetime.

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